Int J Theor Phys (2010) 49: 3118–3130
DOI 10.1007/s10773-010-0390-6

Sequential Product of Quantum Effects: An Overview

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Received: 30 October 2008 / Accepted: 26 May 2010 / Published online: 22 June 2010

Abstract This article presents an overview for the theory of sequential products of quantum effects. We first summarize some of the highlights of this relatively recent field of investigation and then provide some new results. We begin by discussing sequential effect algebras which are effect algebras endowed with a sequential product satisfying certain basic conditions. We then consider sequential products of (discrete) quantum measurements. We next treat transition effect matrices (TEMs) and their associated sequential product. A TEM is a matrix whose entries are effects and whose rows form quantum measurements. We show that TEMs can be employed for the study of quantum Markov chains. Finally, we prove some new results concerning TEMs and vector densities.

Keywords Sequential products · Quantum effects · Quantum Markov chains

1 Introduction

A quantum effect may be thought of as a two-valued quantum measurement that may be unsharp (fuzzy) [1–3, 16–18]. Certain pairs of effects a, b possess an orthosum $a \oplus b$ which corresponds to a disjoint parallel combination and provides a quantum analog to the logical "or" operation. If some natural conditions are placed on $\oplus$, then the set of effects for a quantum system organize into an algebraic structure called an effect algebra E [4, 5, 7]. To study series combinations of effects, we introduce a sequential product $a \circ b$ on E. We think of $a \circ b$ as first performing measurement a and then performing measurement b. The sequential product provides a quantum analog to the logical "and then" operation. Endowing the sequential product with some natural properties we obtain a sequential effect algebra (SEA) [6, 12, 13, 15]. The basic attributes and examples of SEAs are presented. The most important example is a Hilbert space SEA $\mathcal{E}(H)$ over a complex Hilbert space H. The standard sequential product on $\mathcal{E}(H)$ is characterized in Sect. 2.

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Section 3 discusses quantum measurements that can have more than two values and we call them tests. We also consider incomplete quantum measurements called subtests. We then extend our notion of sequential product to obtain sequential products of tests and subtests. We introduce the important concepts of compatibility, coexistence and refinement for tests and subtests. Fundamental results concerning these concepts are summarized.

Section 4 introduces the notion of a transition effect matrix (TEM). A TEM is a square matrix whose entries are effects and whose rows form tests. We define a sequential product of TEMs that generalizes the usual matrix product and note that this is again a TEM. Moreover, the sequential product of TEMs generalizes the previously defined sequential product of tests. A TEM can be employed to define a quantum Markov chain. We briefly investigate the dynamics of quantum Markov chains.

Section 5 provides some new results concerning TEMs and vector densities. In particular, we define projective, invertible, unitary and bistochastic TEMs and prove results about their relationships. We also study singular, and invariant vector densities for a TEM. The paper also includes various examples and open problems.

2 Sequential Effect Algebras

An *effect algebra* is an algebraic system $(E, 0, 1, \oplus)$ where $0, 1 \in E$ and $\oplus$ is a partial binary operation on E satisfying:

(E1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $b \oplus a = a \oplus b$.
(E2) If $a \oplus (b \oplus c)$ is defined, then $(a \oplus b) \oplus c$ is defined and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
(E3) For every $a \in E$ there exists a unique $a' \in E$ such that $a \oplus a' = 1$.
(E4) If $a \oplus 1$ is defined, then $a = 0$.

We define $a \le b$ if there is a c such that $a \oplus c = b$ and we denote this unique c by $c = b \ominus a$. It can be shown that $(E, \le, ')$ is a poset with $0 \le a \le 1$ for all $a \in E$, $a'' = a$ and $a \le b$ implies that $b' \le a'$. Also, $a \oplus b$ is defined if and only if $a \le b'$. We say that a is *sharp* if the greatest lower bound $a \wedge a' = 0$ and denote the set of sharp elements of E by E_S. A map $\phi\colon E \to F$ where E and F are effect algebras is *additive* if $\phi(a) \oplus \phi(b)$ is defined whenever $a \oplus b$ is defined and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$. If $a \le b'$ we say that a and b are *orthogonal* and write $a \perp b$. The partial operation $\oplus$ describes orthogonal parallel combinations of effects. To study series combinations of effects, we introduce a sequential product $a \circ b$ on E. If $a \circ b = b \circ a$ we write $a \mid b$.

A *sequential effect algebra* (SEA) is an algebraic system $(E, 0, 1, \oplus, \circ)$ where $(E, 0, 1, \oplus)$ is an effect algebra and $\circ\colon E \times E \to E$ is a binary operation satisfying:

(S1) $b \to a \circ b$ is additive for all $a \in E$.
(S2) $1 \circ a = a$ for all $a \in E$.
(S3) If $a \circ b = 0$, then $a \mid b$.
(S4) If $a \mid b$, then $a \mid b'$ and $a \circ (b \circ c) = (a \circ b) \circ c$ for all $c \in E$.
(S5) If $c \mid a$ and $c \mid b$, then $c \mid a \circ b$ and $c \mid (a \oplus b)$ whenever $a \perp b$.

We call an operation that satisfied (S1)–(S5) a *sequential product* on E. Condition (S1) states that a sequential product is additive in the second argument. Simple examples in Hilbert space SEAs (to be considered shortly) show that a sequential product need not be additive in the first argument. If $a \mid b$ for all $a, b \in E$, we call E a *commutative* SEA. The next lemma shows that sequential products have desirable properties.

Lemma 2.1 [12] (i) *$a \circ b \le a$ for all $a, b \in E$*. (ii) *If $a \le b$ then $c \circ a \le c \circ b$ for all $c \in E$*. (iii) *$a \in E_S$ if and only if $a \circ a = a$*. (iv) *For $a \in E$, $b \in E_S$, $a \circ b = 0$ if and only if $a \perp b$*.

The next theorem summarizes some of the basic properties of a SEA.

Theorem 2.2 [12] *Let $a \in E$ and $b \in E_S$*. (i) *$a \le b$ if and only if $a \circ b = b \circ a = a$ and $b \le a$ if and only if $a \circ b = b \circ a = b$*. (ii) *If $a \mid b$, then $a \wedge b = a \circ b$*. (iii) *E_S is a sub-effect algebra of E that is an orthomodular poset.*

We say that $a, b \in E$ *coexist* if there exist $c, d, e \in E$ such that $c \oplus d \oplus e$ is defined and $a = c \oplus d$, $b = c \oplus e$. Of course, $c \oplus d \oplus e$ means $(c \oplus d) \oplus e$ but we do not need parentheses because of Condition (E2). We say that $a, b \in E_S$ are *compatible* if there exist mutually orthogonal elements $c, d, e \in E_S$ such that $a = c \vee d$ and $b = c \vee e$.

Theorem 2.3 [12] (i) *If $a \mid b$ then a and b coexist*. (ii) *For $a \in E$, $b \in E_S$, $a \mid b$ if and only if a and b coexist*. (iii) *For $a, b \in E_S$, $a \mid b$ if and only if a and b are compatible.*

Coexistent effects are thought of as effects that can be measured together [1–3, 16–18]. Theorem 2.3 shows that if $a, b \in E_S$ then a and b coexist if and only if $a \mid b$.

We now present examples of the most common SEAs [12, 13]. The set $[0, 1] \subseteq \mathbb{R}$ is an effect algebra where $a \oplus b = a + b$ whenever $a + b \le 1$. Then $[0, 1]$ is a SEA under the unique sequential product $a \circ b = ab$. A Boolean algebra $\mathcal{B}$ is an effect algebra where $a \oplus b = a \vee b$ whenever $a \wedge b = 0$. Then $\mathcal{B}$ is a SEA under the unique sequential product $a \circ b = a \wedge b$. A fuzzy set system $\mathcal{F} = [0, 1]^X$ is an effect algebra where $f \oplus g = f + g$ whenever $f + g \le 1$. Then $\mathcal{F}$ is a SEA under the unique sequential product $f \circ g = fg$. These are examples of commutative SEAs. Our next and most important example is noncommutative. For a complex Hilbert space H we denote the set of bounded linear operators on H by $\mathcal{B}(H)$. We then define

$$\mathcal{E}(H) = \{A \in \mathcal{B}(H) \colon 0 \le A \le I\}$$

It is easy to check that $\mathcal{E}(H)$ is an effect algebra where $A \oplus B = A + B$ whenever $A + B \le I$. Moreover, $\mathcal{E}(H)$ is a SEA under the sequential product

$$A \circ B = A^{1/2} B A^{1/2} \tag{2.1}$$

where $A^{1/2}$ is the unique positive square root of A. We call $\mathcal{E}(H)$ a *Hilbert space* SEA. It is an open problem whether 2.1 is the unique sequential product on $\mathcal{E}(H)$. It has recently been shown that this product is unique if physically motivated conditions are imposed [14].

Theorem 2.4 [14] *The sequential product $A \circ B = A^{1/2} B A^{1/2}$ is the unique binary operation on $\mathcal{E}(H)$ satisfying*:

(1) $A \circ I = I \circ A = A$.
(2) $A \circ (A \circ B) = A^2 \circ B$.
(3) *$A \mapsto A \circ B$ is continuous in the strong operator topology for all $B \in \mathcal{E}(H)$*.
(4) *For every density operator ρ and $A, B \in \mathcal{E}(H)$ we have*

$$\mathrm{tr}\,[(A \circ \rho) B] = \mathrm{tr}\,[\rho (A \circ B)]$$

(5) *If P is a pure state then $(A \circ P)/\mathrm{tr}(A \circ P)$ is a pure state whenever $A \circ P \ne 0$.*

We now give some brief justifications for the conditions (1)–(5). Condition (1) is a mild axiom that holds in any SEA, while (3) is a reasonable axiom that can be justified on physical grounds. Since $A \mid A$, (S4) implies that

$$A \circ (A \circ B) = (A \circ A) \circ B$$

so Condition (2) is essentially the mild axiom that $A \circ A = A^2$ which is certainly physically reasonable. One of the basic postulates of quantum mechanics is that $\mathrm{tr}(\rho A)$ is the probability that the effect A is observed (has value "yes") when the system is in the state ρ. Another basic postulate is that $A \circ \rho / \mathrm{tr}(A \circ \rho)$ is the state ρ conditioned on the effect A being observed (assuming that $A \circ \rho \neq 0$). In this way the sequential product serves a dual role in which $A \circ B$ describes a sequential measurement and $A \circ \rho$ conditions state ρ with a measurement A. Thus, (4) is a duality condition which we can write in the form

$$\mathrm{tr}\,[\rho(A \circ B)] = \mathrm{tr}(A \circ \rho)\frac{\mathrm{tr}[(A \circ \rho)B]}{\mathrm{tr}(A \circ \rho)} \tag{2.2}$$

Now (2.2) is a quantum analogue of Bayes' rule which says that the probability of "A and then B" in the state ρ equals the probability of A in the state ρ times the probability of B in the state ρ conditioned by the occurrence of A. A pure state P is a one-dimensional projection and Condition (5) says that a pure state conditioned by the occurrence of an effect is again a pure state. Again this is a physically reasonable condition.

3 Sequential Products of Measurements

An effect a describes a two-valued (yes–no or 1–0) measurement. More precisely, we should say that the pair $\{a, a'\}$ describes a two-valued measurement. If $\{a, a'\}$ and $\{b, b'\}$ are two-valued measurement in a SEA E, it is natural to define their sequential product to be the four-valued measurement

$$\{a, a'\} \circ \{b, b'\} = \{a \circ b, a \circ b', a' \circ b, a' \circ b'\}$$

In general, a *test* (or *measurement*) is a finite sequence $\{a_i\}$, $a_i \in E$, $i = 1, \ldots, n$, such that $a_1 \oplus \cdots \oplus a_n = 1$. We could also consider infinite sequences, in which case we would work in a σ-SEA [13], but for simplicity we only consider finite tests here. If $\mathcal{A} = \{a_i\}$ is a test, then a_i is the effect observed when $\mathcal{A}$ is performed and the result is the i-th outcome. Thus, a test can be thought of as an n-valued measurement. If $a_i \in E_S$, $i = 1, \ldots, n$, we call $\mathcal{A}$ a *sharp* test. In $\mathcal{E}(H)$, a test corresponds to a (discrete) positive operator-valued measure and a sharp test corresponds to a (discrete) projection-valued measure. We denote the set of tests and sharp tests on E by $\mathcal{T}(E)$ and $\mathcal{S}(E)$, respectively. If $A = \{a_i\}$ with $a_1 \oplus \cdots \oplus a_n \leq 1$, we call $\mathcal{A}$ a *subtest*. A subtest can be thought of as an incomplete test and any subtest can be extended to a test by adjoining $(a_1 \oplus \cdots \oplus a_n)'$. We denote the subtests and sharp subtests by sub-$\mathcal{T}(E)$ and sub-$\mathcal{S}(E)$, respectively.

If $\mathcal{A} = \{a_i\}$, $\mathcal{B} = \{b_i\}$ are subtests we define their *sequential product* by $\mathcal{A} \circ \mathcal{B} = \{a_i \circ b_j\}$. It is easy to see that $\mathcal{A} \circ \mathcal{B} \in$ sub-$\mathcal{T}(E)$ and if $\mathcal{A}, \mathcal{B} \in \mathcal{T}(E)$ then $\mathcal{A} \circ \mathcal{B} \in \mathcal{T}(E)$. For $\mathcal{A}, \mathcal{B} \in$ sub-$\mathcal{T}(E)$ we say that $\mathcal{A}$ and $\mathcal{B}$ are *equivalent* and write $\mathcal{A} \approx \mathcal{B}$ if the nonzero elements of $\mathcal{A}$ are a permutation of the nonzero elements of $\mathcal{B}$. Equivalent subtests can be identified because they are the same except possibly for the order of their outcomes. It is clear that if $\mathcal{A} \approx \mathcal{B}$ then $\mathcal{C} \circ \mathcal{A} \approx \mathcal{C} \circ \mathcal{B}$ and $\mathcal{A} \circ \mathcal{C} \approx \mathcal{B} \circ \mathcal{C}$ for all $\mathcal{C} \in$ sub-$\mathcal{T}(E)$.

We say that $\mathcal{A}=\{a_i\}, \mathcal{B}=\{b_j\}$ are *compatible* if $a_i \mid b_j$ for all i, j. It is clear that if $\mathcal{A}$ and $\mathcal{B}$ are compatible, then $\mathcal{A}\circ\mathcal{B}\approx\mathcal{B}\circ\mathcal{A}$. The converse does not hold. Indeed, $\mathcal{A}\circ\mathcal{A}\approx\mathcal{A}\circ\mathcal{A}$ and yet $\mathcal{A}$ need not be compatible with itself. We conjecture that if $\mathcal{A}\in\mathcal{T}(E)$, $\mathcal{B}\in\mathcal{S}(E)$ and $\mathcal{A}\circ\mathcal{B}\approx\mathcal{B}\circ\mathcal{A}$, then $\mathcal{A}$ and $\mathcal{B}$ are compatible. This conjecture holds in $\mathcal{E}(H)$ [8] but is an open problem in general.

Theorem 3.1 [9] *For $\mathcal{A},\mathcal{B}\in\mathcal{T}(E)$ we have $\mathcal{A}\circ\mathcal{B}\in\mathcal{S}(E)$ if and only if $\mathcal{A},\mathcal{B}\in\mathcal{S}(E)$ and $\mathcal{A}$, $\mathcal{B}$ are compatible.*

The next result shows that a subtest is repeatable if and only if it is sharp.

Theorem 3.2 [9] *For $\mathcal{A}\in$ sub-$\mathcal{T}(E)$ we have $\mathcal{A}\circ\mathcal{A}\approx\mathcal{A}$ if and only if $\mathcal{A}\in$ sub-$\mathcal{S}(E)$.*

For $\mathcal{A},\mathcal{B}\in$ sub-$\mathcal{T}(E)$ with $\mathcal{A}=\{a_i\}$, $\mathcal{B}=\{b_j\}$ we call $\mathcal{A}$ a *refinement* of $\mathcal{B}$ and write $\mathcal{A}\le\mathcal{B}$ if we can adjoin 0s to $\mathcal{A}$ if necessary and organize the elements of $\mathcal{A}$ so that $\mathcal{A}\approx\{a_{ij}\}$ and $b_i=\bigoplus_j a_{ij}$ for every i. For example, $\mathcal{A}\circ\mathcal{B}\le\mathcal{A}$ because $\mathcal{A}\circ\mathcal{B}=\{a_i\circ b_j\}$ and $a_i=\bigoplus_j(a_i\circ b_j)$ for every i. The converse does not hold. That is, $\mathcal{B}\le\mathcal{A}$ does not imply that $\mathcal{B}=\mathcal{A}\circ\mathcal{C}$ for some $\mathcal{C}\in$ sub-$\mathcal{T}(E)$. Strictly speaking we are using equivalence classes in the next theorem because we use $\approx$ instead of equality.

Theorem 3.3 [9] (sub-$\mathcal{T}(E),\le$) *is a poset in which $\mathcal{A}\le\mathcal{B}$ implies $\mathcal{C}\circ\mathcal{A}\le\mathcal{C}\circ\mathcal{B}$ for all $\mathcal{C}\in$ sub-$\mathcal{T}(E)$.*

Theorem 3.4 [9] (i) *If $\mathcal{A}\in$ sub-$\mathcal{S}(E)$, $\mathcal{B}\in$ sub-$\mathcal{T}(E)$ and $\mathcal{B}\le\mathcal{A}$, then $\mathcal{B}\approx\mathcal{A}\circ\mathcal{B}\approx\mathcal{B}\circ\mathcal{A}$.* (ii) *If $\mathcal{A}\in$ sub-$\mathcal{S}(E)$, $\mathcal{B}\in$ sub-$\mathcal{T}(E)$ and $\mathcal{A}\le\mathcal{B}$, then $\mathcal{B}\in$ sub-$\mathcal{S}(E)$ and $\mathcal{A}\approx\mathcal{A}\circ\mathcal{B}=\mathcal{B}\circ\mathcal{A}$.* (iii) *If $\mathcal{A},\mathcal{B}\in\mathcal{T}(E)$ and $\mathcal{A}\circ\mathcal{B}\approx\mathcal{A}$, then $\mathcal{A}$ and $\mathcal{B}$ are compatible, $\mathcal{B}\in\mathcal{S}(E)$ and $\mathcal{A}\le\mathcal{B}$.*

We say that $\mathcal{A},\mathcal{B}\in$ sub-$\mathcal{T}(E)$ *coexist* of they have a common refinement $\mathcal{C}\le\mathcal{A},\mathcal{B}$. Notice, if $\mathcal{A}\circ\mathcal{B}=\mathcal{B}\circ\mathcal{A}$ then $\mathcal{A}\circ\mathcal{B}\le\mathcal{A},\mathcal{B}$ so $\mathcal{A},\mathcal{B}$ coexist. In particular, compatible subtests coexist. It is easy to show that this definition generalizes the definition of coexistence of effects. If $\{a_i\}$, $\{b_j\}$ coexist, then for any fixed i or j, the two-valued tests $\{a_i,a_i'\}$, $\{b_j,b_j'\}$ coexist so a_i, b_j coexist for i, j. The converse does not hold [8].

Theorem 3.5 [9] *Let $\mathcal{A}\in$ sub-$\mathcal{T}(E)$ and $\mathcal{B}\in$ sub-$\mathcal{S}(E)$.* (i) *If $\mathcal{A}$ and $\mathcal{B}$ coexist, then $\mathcal{A},\mathcal{B}$ are compatible.* (ii) *$\mathcal{A}\wedge\mathcal{B}$ exists if and only if $\mathcal{A},\mathcal{B}$ are compatible. In this case $\mathcal{A}\wedge\mathcal{B}=\mathcal{A}\circ\mathcal{B}$.*

Theorem 3.5(ii) shows that (sub-$\mathcal{T}(E),\le$) is not a lattice. The characterization of pairs $\mathcal{A},\mathcal{B}\in$ sub-$\mathcal{T}(E)$ such that $\mathcal{A}\wedge\mathcal{B}$ (or $\mathcal{A}\vee\mathcal{B}$) exist is an open problem.

4 Transition Effect Matrices

If E is an effect algebra, an additive map $s\colon E\to[0,1]\subseteq\mathbb{R}$ such that $s(1)=1$ is called a *state*. If E is a SEA an additive map $\tau\colon E\to[0,\infty]$ is called a *trace* if

(1) $\tau[(a\circ b)\circ c]=\tau[b\circ(a\circ c)]$ for all $a,b,c\in E$,
(2) $\tau(a\circ c)=\tau(b\circ c)<\infty$ for all $c\in E$ implies $a=b$.

We shall assume that τ is a fixed trace on E. Notice by (1) that letting $c = 1$ gives $\tau(a \circ b) = \tau(b \circ a)$ for all $a, b \in E$. An effect $a \in E$ is *trace class* if $\tau(a) < \infty$. We denote the set of trace class effects by $\mathrm{Tr}(E)$. It is easy to check that the usual trace on $\mathcal{E}(H)$ satisfies the previous conditions.

Theorem 4.1 [9] (i) *If* $\mathrm{Tr}(E) \neq \emptyset$, *then* $\tau(0) = 0$. (ii) *If* $a, b \in \mathrm{Tr}(E)$ *and* $a \perp b$, *then* $a \oplus b \in \mathrm{Tr}(E)$. (iii) *If* $b \in \mathrm{Tr}(E)$ *and* $a \leq b$, *then* $a \in \mathrm{Tr}(E)$. (iv) *If* $a \in \mathrm{Tr}(E)$ *and* $b \in E$, *then* $a \circ b, b \circ a \in \mathrm{Tr}(E)$. (v) *If* $\rho \in \mathrm{Tr}(E)$ *with* $\rho \neq 0$, *then* $s(a) = \tau(\rho \circ a)/\tau(\rho)$ *is a state.*

If $\rho \in \mathrm{Tr}(E)$ with $\tau(\rho) = 1$, we call ρ a *density*. A state s is *normal* if there is a $\rho \in \mathrm{Tr}(E)$ such that $s(a) = \tau(\rho \circ a)$ for all $a \in E$. It follows that ρ is a unique density.

A *transition effect matrix* (TEM) is a square matrix $[a_{ij}]$ with $a_{ij} \in E$ and $\bigoplus_j a_{ij} = 1$ for every i. Thus, a TEM is a matrix of effects whose rows are tests. Notice that a TEM on the SEA $[0, 1] \subseteq \mathbb{R}$ is a stochastic matrix. If $A = [a_{ij}]$ and $B = [b_{ij}]$ are TEMs of the same size, we define their product $A \circ B = [c_{ij}]$ by

$$c_{ij} = \bigoplus_k (a_{ik} \circ b_{kj})$$

This generalizes the usual matrix product. It is easy to see that $A \circ B$ is again a TEM. The matrix product of TEMs also generalizes the sequential product of tests. For example, let $\mathcal{A} = \{a, a'\}$ and $\mathcal{B} = \{b, b'\}$ be two-valued tests. Form the TEMs

$$A = \begin{bmatrix} a & a' & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} b & b' & 0 & 0 \\ 0 & 0 & b & b' \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Their product becomes

$$A \circ B = \begin{bmatrix} a \circ b & a \circ b' & a' \circ b & a' \circ b' \\ 0 & 0 & b & b' \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Notice that the first row is $\mathcal{A} \circ \mathcal{B}$. This generalizes to tests with more values.

A *vector density* is a vector $\rho = (\rho_1, \dots, \rho_n)$ where $\rho_i \in \mathrm{Tr}(E)$ and $\sum \tau(\rho_i) = 1$. If $A = [a_{ij}]$ is a TEM and ρ is a vector density of the same size, we define $A * \rho = A^T \circ \rho^T$ to be the vector density given by matrix multiplication

$$(A * \rho)_i = \bigoplus_j (a_{ji} \circ \rho_j)$$

Of course, A^T is the transpose of A. Observe that

$$\tau(\rho) = (\tau(\rho_1), \dots, \tau(\rho_n))$$

is a probability distribution.

A *quantum Markov chain* is a directed graph G in which the edge from vertex i to vertex j is labeled a_{ij} (if there is no such edge, $a_{ij} = 0$) and $A = [a_{ij}]$ forms a TEM. We may think of the vertices of G as sites that a quantum system can occupy and a_{ij} is the effect observed when there is a transition from site i to site j in one time step. Alternatively, the vertices of G could be various configurations for a quantum computer and the effect a_{ij} that the computer evolves from configuration i to configuration j in one time step is governed by a quantum program or algorithm. A quantum Markov chain in the case of the SEA $[0,1] \subseteq \mathbb{R}$ is a classical Markov chain. The system is initially described by a vector density $\rho = (\rho_1, \ldots, \rho_n)$ where $\tau(\rho_i)$ is the probability the system is initially at the site i. For a quantum Markov chain (G, A), $A * \rho$ is the vector density at one time step and the probability distribution at one time step is $\tau(A * \rho)$. At n time steps the vector density is

$$A_{(n)}(\rho) = A * (\cdots A * (A * \rho))$$

and the probability distribution is $\tau[A_{(n)}(\rho)]$. The maps $A_{(n)}$, $n = 1, 2, \ldots,$ are called the *state dynamics*.

For the state dynamics, the vector density evolves and the TEM A is considered fixed. We also have a matrix dynamics in which the TEM evolves and the vector density is considered fixed. This is analogous to the Schrödinger and Heisenberg pictures for quantum dynamics. Although the two types of dynamics are not identical, we shall see that they are statistically equivalent.

The product $A \circ B$ of TEMs is nonassociative in general and when we write $A_n \circ \cdots \circ A_2 \circ A_1$ we mean

$$A_n \circ \cdots \circ \{A_4 \circ [A_3 \circ (A_2 \circ A_1)]\}$$

We define the *n-step* TEM $A^{(n)} = A \circ \cdots \circ A$ (n factors). The maps $\rho \mapsto A^{(n)} * \rho$ are called the *matrix dynamics*. There are examples which show that $A^{(n)} \neq A_{(n)}$, in general [10, 11]. One reason for introducing the matrix dynamics is because the state dynamics $A_{(n)}(\rho)$ depends on ρ, while $A^{(n)}$ is independent of ρ. Thus if a general form of $A^{(n)}$ can be derived, it can be applied to any ρ of the right size.

Lemma 4.2 [10] *If A and B are TEMs of the same size on E, then*

$$\tau[(A \circ B) * \rho] = \tau[B * (A * \rho)]$$

for all ρ of the same size.

The next theorem follows from Lemma 4.2 and mathematical induction. This result shows statistical equivalence for the two types of dynamics.

Theorem 4.3 [10] *If T is a TEM, then $\tau[A^{(n)} * \rho] = \tau[A_{(n)}(\rho)]$ for all ρ of the same size.*

5 Some New Results

This section presents some new results concerning TEMs and vector densities on a SEA E. We first need some preliminary definitions. We say that $\mathrm{Tr}(E)$ is 0-*separating* if $\tau(\rho \circ a) = 0$ for all $\rho \in \mathrm{Tr}(E)$ implies that $a = 0$. It is clear that $\mathrm{Tr}(\mathcal{E}(H))$ is 0-separating for the usual Hilbert space trace. An *effect matrix* is a square matrix $[a_{ij}]$ where $a_{ij} \in E$. If $A = [a_{ij}]$, $B = [b_{ij}]$ are effect matrices such that

$$c_{ij} = \bigoplus_k (a_{ik} \circ b_{kj})$$

is defined for all i, j, we say that $A \circ B$ is *defined* and we set $A \circ B = [c_{ij}]$. It is clear that if A is a TEM, then $A \circ B$ is defined for any effect matrix B. Whenever we write $A \circ B$ we are implicitly assuming that $A \circ B$ is defined. An effect matrix is *sharp* if all its entries are sharp. An effect matrix A is *bistochastic* if both A and its transpose A^T are TEMs. Thus, A is bistochastic if all of its rows and columns form tests. The diagonal effect matrix whose diagonal elements are 1 is denoted by I. We say that an effect matrix A is *invertible* if there exists an effect matrix B such that $A \circ B = B \circ A = I$, A is *unitary* if $A \circ A^T = A^T \circ A = I$ and A is *normal* if $A \circ A^T = A^T \circ A$. Finally, we use the notation $a^2 = a \circ a$ for $a \in E$.

Theorem 5.1 *Let A be a TEM for which $A^T \circ A$ is defined. The following statements are equivalent.* (i) *A is sharp.* (ii) *$A^T \circ A$ is diagonal.* (iii) *The diagonal elements of $A \circ A^T$ are* 1.

Proof We shall show that (ii) and (iii) are both equivalent to A being sharp. Notice that if $\{a_i\} \in \mathcal{S}(E)$ then since $\bigoplus a_i = 1$ we have

$$a_j = a_j \circ 1 = a_j \circ \left(\bigoplus a_i\right) = a_j \oplus \left(\bigoplus_{i \neq j} a_j \circ a_i\right)$$

It follows that $a_j \circ a_i = 0$ for $i \neq j$. Now suppose that A is sharp. Since the rows of A are sharp tests we have that $a_{ij} \circ a_{ik} = 0$ for all i and all $j \neq k$. Hence, if $j \neq k$ we have

$$(A^T \circ A)_{jk} = \bigoplus_j a_{ij} \circ a_{ik} = 0$$

Hence, $A^T \circ A$ is diagonal so (i) implies (ii). Moreover,

$$(A \circ A^T)_{ii} = \bigoplus_j a_{ij} \circ a_{ij} = \bigoplus a_{ij} = 1$$

It follows that (i) implies (iii). Now suppose that $A^T \circ A$ is diagonal. Then for $i \neq j$ we have

$$\bigoplus_k a_{ki} \circ a_{kj} = (A^T \circ A)_{ij} = 0$$

We conclude that $a_{ki} \circ a_{kj} = 0$ for $i \neq j$. Since $\bigoplus_j a_{kj} = 1$ we have

$$a_{ki} = a_{ki} \circ \left(\bigoplus_j a_{kj}\right) = a_{ki}^2$$

for every i and k. Hence, A is sharp so (ii) implies (i). Finally, suppose that $(A \circ a^T)_{ii} = 1$ for all i. We then have

$$\bigoplus_k a_{ik}^2 = (A \circ A^T)_{ii} = 1$$

Since $\bigoplus_k a_{ik} = 1$ and $a_{ik}^2 \leq a_{ik}$ we conclude that

$$\bigoplus_k \left(a_{ik} \ominus a_{ik}^2\right) = 0$$

Hence, $a_{ik} \ominus a_{ik}^2 = 0$ so $a_{ik} = a_{ik}^2$ for all i, k. Therefore, A is sharp and (iii) implies (i). □

It follows from Theorem 5.1 that a unitary effect matrix is sharp and clearly a unitary effect matrix is invertible and normal. The next result provides other relationships for these concepts and the notion of bistochastic.

Theorem 5.2 *If* $\mathrm{Tr}(E)$ *is* 0-*separating*, *then the following statements are equivalent for an effect matrix* A. (i) A *is invertible*. (ii) A *is sharp and bistochastic*. (iii) A *is unitary.* (iv) A *is a sharp*, *normal TEM*.

Proof To show (i) implies (ii), suppose that $A \circ B = B \circ A = I$. Then for $i \neq j$ we have $\bigoplus_k a_{ik} \circ b_{kj} = 0$. Hence, for all k and $i \neq j$ we have

$$a_{ik} \circ b_{kj} = b_{kj} \circ a_{ik} = 0$$

Moreover, $\bigoplus_k a_{ik} \circ b_{ki} = 1$ for all i. Thus, for every $\rho \in \mathrm{Tr}(E)$ and every i we obtain

$$\begin{aligned} \tau(\rho) &= \tau\left[\rho \circ \left(\bigoplus_k a_{ik} \circ b_{ki}\right)\right] = \tau\left[\bigoplus_k \rho \circ (a_{ik} \circ b_{ki})\right] \\ &= \sum_k \tau\left[\rho \circ (a_{ik} \circ b_{ki})\right] = \sum_k \tau\left[(a_{ik} \circ \rho) \circ b_{ki}\right] \\ &\leq \sum_k \tau(a_{ik} \circ \rho) \end{aligned}$$

By symmetry, we have that $\tau(\rho) \leq \sum_k \tau(b_{ik} \circ \rho)$ for every i. Since $a_{ik} \circ b_{kj} = 0$ for every k and $i \neq j$ we have

$$\begin{aligned} \tau\left[\rho \circ (a_{ik} \circ b'_{ki})\right] &= \tau\left[\rho \circ (a_{ik} \ominus a_{ik} \circ b_{ki})\right] = \tau(a_{ik} \circ \rho) - \tau\left[\rho \circ (a_{ik} \circ b_{ki})\right] \\ &\leq \sum_j \tau\left[b_{kj} \circ (a_{ik} \circ \rho)\right] - \tau\left[\rho \circ (a_{ik} \circ b_{ki})\right] \\ &= \sum_j \tau\left[(a_{ik} \circ b_{kj}) \circ \rho\right] - \tau\left[(a_{ik} \circ b_{ki}) \circ \rho\right] \\ &= \sum_{j \neq i} \tau\left[(a_{ik} \circ b_{kj}) \circ \rho\right] = 0 \end{aligned}$$

Since $\mathrm{Tr}(E)$ is 0-separating, $a_{ik} \circ b'_{ki} = 0$ for every i, k. Hence,

$$a_{ik} = a_{ik} \circ b_{ki} = b_{ki} \circ a_{ik} \leq b_{ki}$$

By symmetry, $b_{ki} \leq a_{ik}$ so $b_{ki} = a_{ik}$ for every i, k. We conclude that $B = A^T$. We also have

$$a_{ik} = b_{ki} \circ a_{ik} = a_{ik}^2$$

for every i, k. Hence, A is sharp. Moreover,

$$\bigoplus_k a_{ik} = \bigoplus_k a_{ik} \circ a_{ik} = \bigoplus_k a_{ik} \circ b_{ki} = (A \circ B)_{ii} = 1$$

and by symmetry $\bigoplus_k a_{ki} = 1$ so A is bistochastic. We conclude that (i) implies (ii). To show (ii) implies (iii), suppose A is sharp and bistochastic. Then letting δ_{ij} be the Kroneker delta gives

$$(A \circ A^T)_{ij} = \sum_k a_{ik} \circ a_{jk} = \delta_{ij} \circ \sum_k a_{ik} \circ a_{ik} = \delta_{ij} \circ \sum_k a_{ik} = \delta_{ij} \circ 1$$

and similarly,

$$(A^T \circ A)_{ij} = \sum_k a_{ki} \circ a_{kj} = \delta_{ij} \circ 1$$

Hence, $A \circ A^T = A^T \circ A = I$ so A is unitary. Since (iii) implies (i) is trivial, (i), (ii) and (iii) are equivalent. Now (ii) together with (iii) imply (iv). Also, (iv) implies (ii) because (iv) implies that

$$\begin{aligned}\bigoplus_k a_{kj} &= \bigoplus_k a_{kj} \circ a_{kj} = (A^T \circ A)_{jj} = (A \circ A^T)_{jj} = \bigoplus_i a_{ji} \circ a_{ji} \\ &= \bigoplus_i a_{ji} = 1\end{aligned}$$

so A is bistochastic. Hence, (i), (ii), (iii) and (iv) are equivalent. □

Notice that 0-separating was only used in Theorem 5.2 to show that (i) implies (ii). Thus, all the other implications hold without this condition. We say that a TEM A is *idempotent* if $A \circ A = A$. If (G, A) is a quantum Markov chain with A idempotent, then the distribution of the chain remains invariant after the first time step. Examples of idempotent TEMs in $\mathcal{E}(H)$ are given in [11].

Theorem 5.3 *A sharp TEM* $A = [a_{ij}]$ *is idempotent if and only if* $a_{ij} \le a_{jj}$ *for all* i, j *and* $a_{ik} \circ a_{kj} = 0$ *for all* i *and* $j \ne k$.

Proof Suppose the conditions hold. we then have

$$\begin{aligned}(A \circ A)_{ij} &= \bigoplus_k a_{ik} \circ a_{kj} = a_{ij} \circ a_{jj} \oplus \bigoplus_{k \ne j} a_{ik} \circ a_{kj} \\ &= a_{ij} \circ a_{jj} = a_{ij} = A_{ij}\end{aligned}$$

Hence, $A \circ A = A$ so A is idempotent. Conversely, suppose that A is idempotent. We then have

$$a_{ij} = (A \circ A)_{ij} = \bigoplus_k a_{ik} \circ a_{kj}$$

Hence,

$$\begin{aligned}a_{ij} = a_{ij} \circ a_{ij} = a_{ij} \circ \left(\bigoplus_k a_{ik} \circ a_{kj}\right) = a_{ij} \circ a_{jj} \oplus \bigoplus_{k \ne j} (a_{ij} \circ a_{ik}) \circ a_{kj} \\ = a_{ji} \circ a_{jj}\end{aligned}$$

This implies that $a_{ij} \le a_{jj}$ for all i, j [12]. It follows that

$$\bigoplus_{k \ne j} a_{ik} \circ a_{kj} = 0$$

so that $a_{ik} \circ a_{kj} = 0$ for all i and $j \ne k$. □

A vector density ρ is *invariant* under a TEM A if $A*\rho=\rho$. Of course, if ρ is invariant under A, then the distribution for a corresponding quantum Markov chain in the vector density ρ is constant.

Theorem 5.4 *A vector density $\rho=(\rho_1,\dots,\rho_n)$ is invariant under a sharp TEM $A=[a_{ij}]$ if and only if $\rho_i\circ a_{ii}=a_{ii}\circ\rho_i$ and*

$$\bigoplus_{j\neq i} a_{ji}\circ\rho_j=a'_{ii}\circ\rho_i$$

for all i.

Proof If the conditions hold, then for every i we have

$$\begin{aligned}(A*\rho)_i&=\bigoplus_j a_{ji}\circ\rho_j=a_{ii}\circ\rho_i\oplus\bigoplus_{j\neq i}a_{ji}\circ\rho_j\\&=\rho_i\circ a_{ii}+\rho_i\circ a'_{ii}=\rho_i\end{aligned}$$

Hence, $A*\rho=\rho$ so ρ is invariant under A. Conversely, suppose that ρ is invariant under A. Then for every i we have

$$\rho_i=(A*\rho)_i=a_{ii}\circ\rho_i\oplus\bigoplus_{j\neq i}a_{ji}\circ\rho_j \tag{5.1}$$

Taking the sequential product of both sides with a_{ii} on the left gives

$$\bigoplus_{j\neq i}a_{ii}\circ(a_{ji}\circ\rho_j)=0$$

Therefore, $a_{ii}\circ(a_{ji}\circ\rho_j)=0$ so $a_{ii}\mid a_{ji}\circ\rho_j$ for every i,j with $i\neq j$. Since $a_{ii}\circ\rho_i\le a_{ii}$, it follows that $a_{ii}\mid a_{ii}\circ\rho_j$. By (5.1) we conclude that $a_{ii}\mid\rho_i$ for all i. Applying (5.1) again we have

$$a'_{ii}\circ\rho_i=\rho_i\circ a'_{ii}=\rho_i\ominus\rho_i\circ a_{ii}=\rho_i\ominus a_{ii}\circ\rho_i=\bigoplus_{j\neq i}a_{ji}\circ\rho_j$$ □

Theorems 5.3 and 5.4 involve sharp TEMs. It would be of interest to prove characterizations for arbitrary TEMs.

A 0–1 *TEM* is a TEM whose entries are all 0 or 1. Thus a 0–1 TEM has exactly one 1 in each row and the other entries are 0. A vector density $\rho=(\rho_1,\dots,\rho_n)$ is *singular* if $\rho_j=0$ for $j\neq i$ for some $i=1,\dots,n$. We can represent such a vector density by $\rho_j=\delta_{ji}\circ t$ where $\tau(t)=1$. We say that E is *state* 0*-separating* if $s(a)=0$ for every normal state implies $a=0$. Of course, if E is state 0-separating, then $\operatorname{Tr}(E)$ is 0-separating.

Theorem 5.5 (i) *Let ρ be the singular vector density $\rho_j=\delta_{jk}\circ t$. Then ρ is invariant under the TEM $A=[a_{ij}]$ if and only if $a_{kk}\circ t=t$.* (ii) *Assume that E is state* 0*-separating. Then a TEM A on E takes every singular vector density to a singular vector density if and only if A is a* 0–1 *TEM.*

Proof (i) If ρ is invariant under A, then

$$\delta_{ik} \circ t = \rho_i = (A * \rho)_i = \bigoplus_j a_{ji} \circ \rho_j = a_{ki} \circ t$$

Hence, $a_{kk} \circ t = t$. Conversely, if $a_{kk} \circ t = t$ we have that [12] $a_{kk} \mid t$. Therefore,

$$t = \bigoplus_j t \circ a_{jk} = t \oplus \bigoplus_{j \neq k} t \circ a_{kj}$$

But this implies that $t \circ a_{kj} = 0$ for $j \neq k$. Hence, $a_{kj} \circ t = 0$ for $j \neq k$. We conclude that

$$(A * \rho)_i = \bigoplus_j a_{ji} \circ \rho_j = a_{ki} \circ t = \delta_{ik} \circ t = \rho_i$$

Thus, ρ is invariant under A. (ii) Let $\rho_j = \delta_{rj} \circ t_1$ and $\omega_j = \delta_{sj} \circ t_2$ be singular vector densities and suppose $A * \rho = \omega$. Then for every i we have

$$(A * \rho)_i = \bigoplus_k a_{ki} \circ \rho_k = a_{ri} \circ t_1 = \omega_i = \delta_{si} \circ t_2$$

Hence, $a_{ri} \circ t_1 = 0$ if $i \neq s$. Since this holds for every $t_1 \in \mathrm{Tr}(E)$ with $\tau(t_1) \neq 0$ and E is state 0-separating, we conclude that $a_{ri} = 0$ for every $i \neq s$. Therefore, $a_{ri} = \delta_{ri} \circ 1$. Since r is arbitrary, A is a 0–1 TEM. Conversely, suppose A is a 0–1 TEM. If $\rho_j = \delta_{rj} \circ t$ is a singular vector density, then

$$(A * \rho)_i = a_{ri} \circ t = t$$

for some i and $(A \circ \rho)_j = 0$ for $j \neq i$. Hence, $(A * \rho)_j = \delta_{ij} \circ t$. Therefore, $A * \rho = \omega$ where $\omega_j = \delta_{ij} \circ t$. Hence, A takes singular vector densities to singular vector densities. □

Acknowledgement The author thanks the referee for pointing out an error in the original manuscript.

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